

# Regularity of solutions to higher-order integrals of the calculus of variations

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## Abstract

We obtain new regularity conditions for problems of calculus of variations with higher-order derivatives. As a corollary, we get non-occurrence of the Lavrentiev phenomenon. Our main regularity result asserts that autonomous integral functionals with a Lagrangian having coercive partial derivatives with respect to the higher-order derivatives admit only minimizers with essentially bounded derivatives.

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## 1 Introduction and preliminaries

Let  $\mathcal{L}(t, x^0, \dots, x^m)$  be a given  $C^1([a, b] \times \mathbb{R}^{(m+1) \times n})$  real function. The problem of the calculus of variations with high-order derivatives consists in minimizing an integral functional

$$I[x(\cdot)] = \int_a^b \mathcal{L}(t, x(t), \dot{x}(t), \dots, x^{(m)}(t)) dt \quad (P_m)$$

over a certain class  $\mathcal{X}$  of functions  $x : [a, b] \rightarrow \mathbb{R}^n$  satisfying the boundary conditions

$$x(a) = x_a^0, x(b) = x_b^0, \dots, x^{(m-1)}(a) = x_a^{m-1}, x^{(m-1)}(b) = x_b^{m-1}. \quad (1)$$

Often it is convenient to write  $x^{(1)} = x'$ ,  $x^{(2)} = x''$ , and sometimes we revert to the standard notation used in mechanics:  $x' = \dot{x}$ ,  $x'' = \ddot{x}$ . Such problems

arise, for instance, in connection with the theory of beams and rods [22]. Further, many problems in the calculus of variations with higher-order derivatives describe important optimal control problems with linear dynamics [20].

Regularity theory for optimal control problems is a fertile field of research and a source of many challenging mathematical issues and interesting applications [6, 25, 26, 12]. The essential points in the theory are: (i) existence of minimizers and (ii) necessary optimality conditions to identify those minimizers.

The first systematic approach to existence theory was introduced by Tonelli in 1915 [23], who showed that existence of minimizers is guaranteed in the Sobolev space  $W_m^m$  of absolutely continuous functions. The direct method of Tonelli proceeds in three steps: (i) regularity, and convexity with respect to the highest-derivative of the Lagrangian  $\mathcal{L}$ , guarantees lower semi-continuity; (ii) the coercivity condition (the Lagrangian  $\mathcal{L}$  must grow faster than a linear function) implies that minimizing sequences lie in a compact set; (iii) thus, by the compactness principle, one gets directly from (i) and (ii) the existence of minimizers for the problem  $(P_m)$ . Typically, Tonelli's existence theorem for  $(P_m)$  is formulated as follows.<sup>1</sup>

**Theorem 1.1.** (see e.g. [6, 10, 27]) *Under hypotheses (H1)-(H3) on the Lagrangian  $\mathcal{L}$ ,*

(H1)  $\mathcal{L}(t, x^0, \dots, x^m)$  *is locally Lipschitz in*  $(t, x^0, \dots, x^m)$ ;

(H2)  $\mathcal{L}(t, x^0, \dots, x^m)$  *is convex as a function of the last argument*  $x^m$ ;

(H3)  $\mathcal{L}(t, x^0, \dots, x^m)$  *is coercive in*  $x^m$ , *i.e.*  $\exists \theta : [0, \infty) \rightarrow \mathbb{R}$  *such that*

$$\lim_{r \rightarrow \infty} \frac{\theta(r)}{r} = +\infty, \\ \mathcal{L}(t, x^0, \dots, x^m) \geq \theta(|x^m|) \text{ for all } (t, x^0, \dots, x^m),$$

*there exists a minimizer to problem  $(P_m)$  in the class  $W_m^m$ .*

The main necessary condition in optimal control is the famous Pontryagin maximum principle, which includes all the classical necessary optimality conditions of the calculus of variations [17]. It turns out that the hypotheses (H1)-(H3) do not assure the applicability of the necessary optimality conditions, being required more regularity on the class of admissible functions [1]. For  $(P_m)$ , the Pontryagin maximum principle [17] is established assuming  $x \in W_m^\infty \subset W_m^m$ .

In the case  $m = 1$ , extra information about the minimizers was proved, for the first time, by Tonelli himself [23]. Tonelli established that, under the hypotheses (H2) and (H3) of convexity and coercivity, the minimizers  $x$  have the property that  $\dot{x}$  is locally essentially bounded on an open subset  $\Omega \subset [a, b]$  of full measure. If the following Tonelli-Morrey regularity condition [9, 20, 7]

$$\left| \frac{\partial \mathcal{L}}{\partial x} \right| + \left| \frac{\partial \mathcal{L}}{\partial \dot{x}} \right| \leq c|\mathcal{L}| + r, \quad (2)$$

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<sup>1</sup>In our context (H1) is trivially satisfied since we are assuming  $\mathcal{L}$  to be a  $C^1$  function. It is customary to choose the function  $\theta$  in hypothesis (H3) as  $\theta(r) = ar^2 + b$  for some strictly positive constants  $a$  and  $b$ . We then say that  $\mathcal{L}$  is *quadratically coercive*.

is satisfied for some constants  $c$  and  $r$ ,  $c > 0$ , then  $\Omega = [a, b]$  ( $\dot{x}(t)$  is essentially bounded in all points  $t$  of  $[a, b]$ , i.e.  $x \in W_1^\infty$ ), and the Pontryagin maximum principle, or the necessary condition of Euler-Lagrange, hold. Since L. Tonelli and C. B. Morrey [15], several Lipschitzian regularity conditions were obtained for the problem  $(P_m)$  with  $m = 1$ : S. Bernstein [2], for the scalar case  $n = 1$ , F. H. Clarke and R. B. Vinter [8], for the vectorial case  $n > 1$ , obtained the condition

$$\left| \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} \right)^{-1} \left( \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial^2 \mathcal{L}}{\partial \dot{x} \partial t} - \frac{\partial^2 \mathcal{L}}{\partial \dot{x} \partial x} \dot{x} \right) \right| \leq c \left( |\dot{x}|^3 + 1 \right), \quad \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} > 0;$$

F. H. Clarke and R. B. Vinter [8] the regularity conditions

$$\left| \frac{\partial \mathcal{L}}{\partial t} \right| \leq c |\mathcal{L}| + k(t), \quad k(\cdot) \in L_1, \quad (3)$$

and

$$\left| \frac{\partial \mathcal{L}}{\partial x} \right| \leq c |\mathcal{L}| + k(t) \left| \frac{\partial \mathcal{L}}{\partial \dot{x}} \right| + m(t), \quad k(\cdot), m(\cdot) \in L_1;$$

and A. V. Sarychev and D. F. M. Torres [19] the condition

$$\left( \left| \frac{\partial \mathcal{L}}{\partial t} \right| + \left| \frac{\partial \mathcal{L}}{\partial x} \right| \right) |\dot{x}|^\mu \leq \gamma \mathcal{L}^\beta + \eta, \quad \gamma > 0, \beta < 2, \mu \geq \max \{ \beta - 1, -1 \}. \quad (4)$$

Lipschitzian regularity theory for the problem of the calculus of variations with  $m = 1$  is now a vast discipline (see e.g. [4, 11, 3, 16, 26] and references therein). Results for  $m > 1$  are scarcer: we are aware of the results in [10, 19, 24]. In 1997 A.V. Sarychev [18] proved that the second-order problems of the calculus of variations may show new phenomena non-present in the first-order case: under the hypotheses (H1)-(H3) of Tonelli's existence theory, autonomous problems  $(P_m)$  with  $m = 2$  may present the Lavrentiev phenomenon [14]. This is not a possibility for  $m = 1$ , as shown by the Lipschitzian regularity condition (3). Sarychev's result was recently extended by A. Ferriero [13] for the case  $m > 2$ . It is also shown in [13] that, under some standard hypotheses, the problems of the calculus of variations  $(P_m)$  with Lagrangians only depending on two consecutive derivatives  $x^{(\gamma)}$  and  $x^{(\gamma+1)}$ ,  $\gamma \geq 0$ , do not exhibit the Lavrentiev phenomenon for any boundary conditions (1) (for  $m = 1$ , this follows immediately from (3)). In the case in which the Lagrangian only depends on the higher-order derivative  $x^{(m)}$ , it is possible to prove more [19, Corollary 2]: when  $\mathcal{L} = \mathcal{L}(x^{(m)})$ , all the minimizers predicted by the existence theory belong to the space  $W_m^\infty \subset W_m^m$  and satisfy the Pontryagin maximum principle (regularity). As to whether this is the case or not for Ferriero's problem with Lagrangians only depending on consecutive derivatives  $x^{(\gamma)}$  and  $x^{(\gamma+1)}$ , seems to be an open question.

The results of Sarychev [18] and Ferriero [13] on the Lavrentiev phenomenon show that the problems of the calculus of variations with higher-order derivatives are richer than the problems with  $m = 1$ , but also show, in our opinion, that the regularity theory for higher-order problems is underdeveloped. One can

say that the Lipschitzian regularity conditions found in the literature for the higher-order problems of the calculus of variations are a generalization of the above mentioned conditions for  $m = 1$ : [10] generalizes [8] for  $m > 1$ , [19] generalizes (4) for problems of optimal control with control-affine dynamics, [24] generalizes (2) for optimal control problems with more general nonlinear dynamics. To the best of our knowledge, there exist no regularity conditions for the higher-order problems of the calculus of variations of a different type from those also obtained (also valid) for the first-order problems. We prove here a new regularity condition which is of a different nature than those appearing for the first-order problems. The results of the paper extend those found in [21], covering problems of the calculus of variations with derivatives of higher order than two. While existence follows by imposing coercivity to the Lagrangian  $\mathcal{L}$  (hypothesis (H3)), we prove (cf. Theorem 4.1) that for the autonomous high-order problems of the calculus of variations, regularity follows by imposing a superlinear condition with respect to the sum of the partial derivatives  $\frac{\partial \mathcal{L}}{\partial x_i^{(m)}}$  of the Lagrangian. We observe that our condition is intrinsic to the higher-order problems: for autonomous problems of the calculus of variations with  $m = 1$  (3) is trivially satisfied and no coercivity on the partial derivatives  $\frac{\partial \mathcal{L}}{\partial x}$  are needed. Our condition is, however, necessary, as a consequence of Sarychev's results [18].

## 2 Outline of the paper and hypotheses

In Section 3 we establish a generalized integral form of duBois-Reymond necessary condition, valid in the class  $\mathcal{X} = W_m^m$  (we recall that the optimal solutions  $x$  may have unbounded derivatives). In Section 4.1 we make use of our duBois-Reymond necessary condition to obtain regularity conditions under which all the minimizers of  $(P_m)$  are in  $W_m^\infty \subset W_m^m$  and thus satisfy the classical necessary conditions. Then, in Section 4.2, arguments analogous to those used to prove the duBois-Reymond necessary condition in §3 are used to prove an Euler-Lagrange necessary condition valid for non-regular minimizers  $W_m^m$  not in  $W_m^\infty$ . In general terms, one can say that the techniques used here are an extension of those appearing in [5] and [21].

In the sequel we shall assume the following hypotheses, where  $x(\cdot) \in W_m^m$  is the minimizer under consideration:

- (S<sub>0</sub>) There exists a continuous function  $S \geq 0$ ,  $(t, x^0, x^1, \dots, x^m) \in \mathbb{R}^{1+(m+1) \times n}$ , and some  $\delta > 0$ , such that  $t \rightarrow S(t, x^0(t), \dots, x^m(t))$  is  $L^{m'}$ -integrable in  $[a, b]$ ,  $m'$  being the Holder conjugate of  $m$  ( $\frac{1}{m} + \frac{1}{m'} = 1$ ) and

$$\left| \frac{\partial \mathcal{L}}{\partial t}(\tau, x^0, x^1, \dots, x^m) \right| \leq S(t, x^0, x^1, \dots, x^m),$$

for any  $t \in [a, b]$ ,  $|\tau - t| < \delta$ ,  $x^i = x^i(t)$ ,  $0 \leq i \leq m$ .

( $S_i$ ) There exists a nonnegative continuous function  $G$ , and some  $\delta > 0$ , such that  $t \rightarrow G(t, x^0(t), x^1(t), \dots, x^m(t))$  is  $L^{m'}$ -integrable on  $[a, b]$ , and

$$\left| \frac{\partial \mathcal{L}}{\partial x_i^{(k)}}(t, x^0, x^1, \dots, x^{k-1}, y, x^{k+1}, \dots, x^m) \right| \leq G(t, x^0, \dots, x^m),$$

for any  $t \in [a, b]$ ,  $x, x^1, \dots, x^m \in \mathbb{R}^n$ ,  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ ,  $y_j = x_j^k(t)$  for  $j \neq i$ ,  $|y_i - x_i^k(t)| \leq \delta$ ,  $i = 1, \dots, n$  and  $k = 0, 1, 2, \dots, m$ , where  $x_i^k(t)$  is the  $i^{th}$  component of the  $k^{th}$  vector with the convention  $x_i^0(t) = x_i(t)$ .

*Remark 2.1.* Hypothesis ( $S_0$ ) is certainly verified if  $\mathcal{L}$  does not depend on  $t$ : ( $S_0$ ) holds trivially in the autonomous case. Conditions ( $S_i$ ),  $i = 0, \dots, n$ , are needed in the proof of Theorems 3.1 and 4.3 to justify the usual rule of differentiation under the sign of an integral.

### 3 Generalized duBois-Reymond equation

In this section we prove an integral form of the duBois Reymond equation (equality (5) of Theorem 3.1 below). For this, we consider an arbitrary change of the independent variable  $t$ . Let  $s$  be the arc length parameter on the curve  $C_0 : x = x(t)$ ,  $a \leq t \leq b$ , so that the Jordan length of  $C_0$  is  $s(t) = \int_a^t \sqrt{1 + (x'(\tau))^2} d\tau$  with  $s(a) = 0$ ,  $s(b) = l$  and  $s(t)$  is absolutely continuous with  $s'(t) \geq 1$  a.e. Thus  $s(t)$  and its inverse  $t(s)$ ,  $0 \leq s \leq l$ , are absolutely continuous with  $t'(s) > 0$  a.e. in  $[0, l]$ . If  $X(s) = x(t(s))$ ,  $0 \leq s \leq l$ , then  $t(s)$  and  $X(s)$  are Lipschitzian of constant one in  $[0, l]$ . By the usual change of variable,

$$\begin{aligned} I[x] &= \int_a^b \mathcal{L} \left( t, x(t), \dot{x}(t), \dots, x^{(m)}(t) \right) dt \\ &= \int_0^l \mathcal{L} \left( t(s), X(s), \frac{X'(s)}{t'(s)}, \sum_{i=1}^2 P_{i2}(t', t'') X^{(i)}, \right. \\ &\quad \left. \dots, \sum_{i=1}^m P_{im}(t', t'', \dots, t^{(m)}) X^{(i)} \right) t'(s) ds, \end{aligned}$$

where  $P_{ik}$ ,  $1 \leq k \leq m$ , are functions on  $(t', t'', \dots, t^{(k)})$ , obtained by differentiating  $X(s)$   $k$ -times and replacing the derivatives  $x^i(t(s))$  by  $X^i(s)$ ,  $i = 1, \dots, k-1$ . Setting

$$\begin{aligned} F(t, x, t', x', t'', x'', \dots, t^{(m)}, x^{(m)}) \\ = \mathcal{L} \left( t, x, \frac{x'}{t'}, \sum_{i=1}^2 P_{i2}(t', t'') x^{(i)}, \dots, \sum_{i=1}^k P_{ik}(t', t'', \dots, t^{(k)}) x^{(i)}, \right. \\ \left. \dots \sum_{i=1}^m P_{im}(t', t'', \dots, t^{(m)}) x^{(i)} \right) t', \end{aligned}$$

then we have:

$$I[x] = J[C] = J[X] = \int_0^l F\left(t(s), X(s), t'(s), X'(s), \dots, t^{(m)}(s), X^{(m)}(s)\right) ds.$$

Thus, after reparameterization by length, the cost functional can be considered as a functional  $J[C]$  in the space of curves, rather than a functional in the space of functions  $W_m^m$ .

*Remark 3.1.* For  $m = 2$ , we have  $F(t, x, t', x', t'', x'') = \mathcal{L}\left(t, x, \frac{x'}{t'}, \frac{1}{t'^2}x'' - \frac{t''}{t'^3}x'\right) t'$ .

The following necessary condition will be useful to prove our regularity theorem (Theorem 4.1).

**Theorem 3.1.** *Under hypotheses  $(S_i)_{0 \leq i \leq n}$ , if  $x(\cdot) \in W_m^m$  is a minimizer of problem  $(P_m)$ , then the following integral form of duBois-Reymond necessary condition holds:*

$$\phi_0(s) = \frac{\partial F}{\partial t^{(m)}} + \sum_{i=1}^m (-1)^{m-i+1} \int_0^s \int_0^{\tau_1} \dots \int_0^{\tau_{m-i}} \frac{\partial F}{\partial t^{(i-1)}} d\sigma d\tau_{m-i} \dots d\tau_1 = c_0, \quad (5)$$

where  $0 \leq \tau_i \leq s \leq l$ ,  $c_0$  is a constant, and functions  $\frac{\partial F}{\partial t^{(i)}}$ ,  $1 \leq i \leq m$ , are evaluated at  $(t(s), X(s), t'(s), X'(s), \dots, t^{(m)}(s), X^{(m)}(s))$ .

*Remark 3.2.* For  $m = 2$  (5) takes the following form:

$$\phi_0(s) = \frac{\partial F}{\partial t''} - \int_0^s \frac{\partial F}{\partial t'} + \int_0^s \int_0^\tau \frac{\partial F}{\partial t} = c_0, \quad 0 \leq \tau \leq s \leq l.$$

*Proof.* It is to be noted that  $(t(s), X(s), t'(s), \dots, X^{(m)}(s), t^{(m)}(s))$  may not exist in a set of null-measure of all  $s$ . The proof is done by contradiction. Suppose that (5) is not true. Then, there exist constants  $d_1 < d_2$  and disjoint sets  $E_1^*$  and  $E_2^*$  of non-zero measure such that

$$\begin{aligned} \phi_0(s) &\leq d_1 \text{ for } s \in E_1^*, \\ \phi_0(s) &\geq d_2 \text{ for } s \in E_2^*, \end{aligned}$$

while  $t'(s) > 0$  a.e in  $[0, l]$ . Hence, there exist some constant  $k > 0$  and two subsets  $E_1, E_2$  of positive measure of  $E_1^*, E_2^*$ , such that

$$t'(s) \geq k > 0, \quad \phi_0(s) \leq d_1 \quad \text{for } s \in E_1, \quad |E_1| > 0, \quad (6)$$

$$t'(s) \geq k > 0, \quad \phi_0(s) \geq d_2 \quad \text{for } s \in E_2, \quad |E_2| > 0. \quad (7)$$

Let us consider

$$\psi(s) = \int_0^s \int_0^{\tau_1} \dots \int_0^{\tau_{m-1}} \{|E_2| \chi_1 - |E_1| \chi_2\} d\sigma d\tau_{m-1} \dots d\tau_1,$$

$0 \leq \tau_i \leq s \leq l$ ,  $1 \leq i \leq m-1$ , where  $\chi_j$  denotes the indicator function defined by

$$\chi_j(s) = \begin{cases} 1 & \text{for } s \in E_j, \\ 0 & \text{for } s \in [0, l] \setminus E_j, \end{cases} \quad j = 1, 2 \text{ and } 0 \leq s \leq l.$$

We have that  $\psi^{(m-1)}$  is an absolutely continuous function in  $[0, l]$  with  $\psi^{(m-1)}(0) = \psi^{(m-1)}(l) = 0$ . Moreover,

$$\psi^{(m)}(s) = \begin{cases} -|E_1| & a.e. \quad s \in E_2, \\ |E_2| & a.e. \quad s \in E_1, \\ 0 & a.e. \quad s \in [0, l] - E_1 \cup E_2. \end{cases}$$

We also define  $C_\alpha : t = t_\alpha(s)$ ,  $x = X_\alpha(s)$ ,  $0 \leq s \leq l$ , by setting

$$t_\alpha(s) = t(s) + \sum_{i=1}^m \alpha^i \psi^{(i-1)}(s),$$

$$X_\alpha(s) = X(s), \quad 0 \leq s \leq l, \quad |\alpha| \leq 1.$$

Let  $\rho > 0$  be chosen in such a way that  $t, \tau \in [a, b]$  and  $|t - \tau| < \rho$  imply  $|x(t) - x(\tau)| \leq \delta$ , where  $\delta$  is the constant in condition  $(S_0)$ . We now choose  $\alpha$  small enough,  $|\alpha| \leq \alpha_0$ , to give  $t'_\alpha(s) > 0$  for  $s \in E_1 \cup E_2$ , and  $C_\alpha$  has an absolutely continuous representation  $x = x_\alpha(t)$ ,  $a \leq t \leq b$ . We also have  $|t_\alpha(s) - t(s)| < \rho$ . Hence  $|x_\alpha(t) - x(t)| = |x(t_\alpha(s)) - x(t(s))| < \delta$  and we conclude that  $J[C_\alpha] \geq J[C]$ . On the other hand, by setting  $\phi(\alpha, s) = F(t, X, t', X', \dots, t^{(m)}, X^{(m)})$ , we have by differentiation that

$$\left. \frac{\partial \phi}{\partial \alpha} \right|_{\alpha=0} = \frac{\partial F}{\partial t} \psi + \frac{\partial F}{\partial t'} \psi' + \dots + \frac{\partial F}{\partial t^{(m)}} \psi^{(m)},$$

where

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial \mathcal{L}}{\partial t} t', \\ &\vdots \\ \frac{\partial F}{\partial t^{(k)}} &= t' \frac{\partial \mathcal{L}}{\partial x^{(k)}} \sum_{i=1}^k \frac{\partial P_{ik}}{\partial t^{(k)}} x^{(i)} + \dots + t' \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^k \frac{\partial P_{im}}{\partial t^{(k)}} x^{(i)} \\ &\vdots \\ \frac{\partial F}{\partial t^{(m-1)}} &= t' \frac{\partial \mathcal{L}}{\partial x^{(m-1)}} \sum_{i=1}^{m-1} \frac{\partial P_{i(m-1)}}{\partial t^{(m-1)}} x^{(i)} + t' \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m \frac{\partial P_{im}}{\partial t^{(m-1)}} x^{(i)} \\ \frac{\partial F}{\partial t^{(m)}} &= t' \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m \frac{\partial P_{im}}{\partial t^{(m)}} x^{(i)}. \end{aligned} \tag{8}$$

By hypotheses  $(S_i)_{0 \leq i \leq n}$ , both absolute value of terms  $\frac{\partial F}{\partial t} \psi, \frac{\partial F}{\partial t'} \psi', \dots, \frac{\partial F}{\partial t^{(m)}} \psi^{(m)}$  are bounded in  $E_1 \cup E_2$  by a fixed function which is  $L$ -integrable in  $[0, l]$ . Then,

we can differentiate under the sign of the integral to obtain:

$$0 = \left. \frac{\partial J(C_\alpha)}{\partial \alpha} \right|_{\alpha=0} = \int_0^l \left( \frac{\partial F}{\partial t} \psi + \frac{\partial F}{\partial t'} \psi' + \dots, \frac{\partial F}{\partial t^{(m)}} \psi^{(m)} \right) ds.$$

Integration by parts, and using (6)–(7), yields

$$\begin{aligned} 0 &= \int_0^l \phi_0(s) \psi^{(m)} ds = \int_{E_1} \phi_0(s) \psi^{(m)} ds + \int_{E_2} \phi_0(s) \psi^{(m)} ds \\ &\leq |E_2| |E_2| (d_1 - d_2) < 0 \end{aligned}$$

which is a contradiction. Equality (5) is proved.  $\square$

## 4 Main results

In §4.1 we obtain a new regularity result which implies the validity of the classical Euler-Lagrange necessary condition. In §4.2 a new Euler-Lagrange necessary condition is proved which is valid both for regular and non-regular minimizers.

### 4.1 Regularity for autonomous problems

We shall present now a regularity result for  $(P_m)$  under certain additional requirements on the Lagrangian  $\mathcal{L}$ .

**Theorem 4.1.** *In addition to the hypotheses  $(S_i)_{0 \leq i \leq n}$ , let us consider the autonomous problem  $(P_m)$ , i.e. let us assume that  $\mathcal{L}$  does not depend on  $t$ :  $\mathcal{L} = \mathcal{L}(x, \dot{x}, \dots, x^{(m)})$ . If  $\frac{\partial \mathcal{L}}{\partial x^{(m)}}$  is superlinear with respect to the sum of the derivatives  $x^{(i)}, i = 1, \dots, m$ , i.e. there exist constants  $a > 0$  and  $b > 0$  such that*

$$a \sum_{i=1}^m |x^{(i)}| + b \leq \left| \frac{\partial \mathcal{L}}{\partial x^{(m)}}(x, \dot{x}, \dots, x^{(m)}) \right|, \quad (9)$$

*then every minimizer  $x \in W_m^m$  of the problem is on  $W_m^\infty$ .*

**Corollary 4.2.** *Under the hypotheses of Theorem 4.1, the autonomous problem of the calculus of variations with higher-order derivatives do not admit the Lavrentiev gap  $W_m^m - W_m^\infty$ :*

$$\begin{aligned} \inf_{x(\cdot) \in W_m^m} \int_a^b \mathcal{L}(x(t), \dot{x}(t), \dots, x^{(m)}(t)) dt \\ = \inf_{x(\cdot) \in W_m^\infty} \int_a^b \mathcal{L}(x(t), \dot{x}(t), \dots, x^{(m)}(t)) dt. \end{aligned}$$



*Proof.* Using (5), (8), the fact that we consider the autonomous case, and applying Holder's inequality, we get

$$\begin{aligned} \left| t' \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m \frac{\partial P_{im}}{\partial t^{(m)}} x^{(i)} \right| &\leq c_0 + c_1 + \left| t' \int_0^s \sum_{i=1}^m \frac{\partial P_{im}}{\partial t^{(m-1)}} x^{(i)} \frac{\partial \mathcal{L}}{\partial x^{(m)}} \right| \\ &+ \left| \int_0^s t' \sum_{i=1}^{m-1} \frac{\partial P_{i(m-1)}}{\partial t^{(m-1)}} x^{(i)} \frac{\partial \mathcal{L}}{\partial x^{(m-1)}} \right|, \end{aligned}$$

for positive constants  $c_0$  and  $c_1$ . Therefore, with the aid of the condition  $(S_i)$ , we have

$$\left| \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m x^{(i)} \right| \leq c_3 + c_4 \int_0^s \left| \sum_{i=1}^m x^{(i)} \frac{\partial \mathcal{L}}{\partial x^{(m)}} \right|$$

where  $c_3$  and  $c_4$  are positive constants. Then, using the fact that  $\mathcal{L} \in C^1$ ,  $\frac{\partial \mathcal{L}}{\partial \dot{x}}, \dots, \frac{\partial \mathcal{L}}{\partial x^{(m)}} \in L^m$  and  $x \in W_m^m$  (in other terms,  $x, \dot{x}, \dots, x^{(m)} \in L^m$ ), it follows by the Gronwall lemma that  $\frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m x^{(i)}$  satisfies a condition of the form

$$\left| \frac{\partial \mathcal{L}}{\partial x^{(m)}} \sum_{i=1}^m x^{(i)} \right| \leq c_5$$

for a certain positive constant  $c_5$ . Besides, since  $\frac{\partial \mathcal{L}}{\partial x^{(m)}}$  verifies (9), we have

$$\left( a \sum_{i=1}^m |x^{(i)}| + b \right) \left| \sum_{i=1}^m x^{(i)} \right| \leq c_5 \quad (b > 0).$$

Therefore,  $\sum_{i=1}^m |x^{(i)}|$  and  $\frac{\partial \mathcal{L}}{\partial x}$  are uniformly bounded. This implies that  $|x^{(i)}|$ ,  $1 \leq i \leq m$ , are essentially bounded.  $\square$

## 4.2 Generalized Euler-Lagrange equation

If Theorem 4.1 holds, then one can use the classical Euler-Lagrange equation to obtain the minimizers. If this is not the case, i.e. Theorem 4.1 does not hold, it may happen that minimizers fail to satisfy the standard Euler-Lagrange equations. Next we give a generalized Euler-Lagrange necessary condition which is valid in the class of functions  $W_m^m$  where existence is proved.

**Theorem 4.3.** *Under the hypotheses  $(S_i)_{0 \leq i \leq n}$ , if  $x(\cdot) \in W_m^m$  is a minimizer of problem  $(P_m)$ , then we have the following integral form of the Euler-Lagrange equations:*

$$\phi_i(s) = \frac{\partial F}{\partial x_i^{(m)}} + \sum_{j=1}^m (-1)^{m-j+1} \int_0^s \int_0^{\tau_1} \dots \int_0^{\tau_{m-j}} \frac{\partial F}{\partial x_i^{(j-1)}} d\sigma d\tau_{m-j} \dots d\tau_1 = c_i, \quad (10)$$

where functions  $\frac{\partial F}{\partial x_i^{(j)}}$  are evaluated at  $(t(s), X(s), t'(s), X'(s), \dots, t^{(m)}(s), X^{(m)}(s))$ ,  $c_i$  denote constants,  $i = 1, \dots, n$ , and

$$\begin{aligned} \frac{\partial F}{\partial x_i} &= t' \frac{\partial \mathcal{L}}{\partial x_i} \\ \frac{\partial F}{\partial \dot{x}_i} &= \frac{\partial \mathcal{L}}{\partial \dot{x}_i} + t' \left( P_{12} \frac{\partial \mathcal{L}}{\partial \ddot{x}_i} + \dots + P_{1m} \frac{\partial \mathcal{L}}{\partial x_i^{(m)}} \right), \\ &\vdots \\ \frac{\partial F}{\partial x_i^{(k)}} &= t' \left( P_{kk} \frac{\partial \mathcal{L}}{\partial x_i^{(k)}} + P_{k(k+1)} \frac{\partial \mathcal{L}}{\partial x_i^{(k+1)}} + \dots + P_{km} \frac{\partial \mathcal{L}}{\partial x_i^{(m)}} \right), \\ &\vdots \\ \frac{\partial F}{\partial x_i^{(m)}} &= t' P_{mm} \frac{\partial \mathcal{L}}{\partial x_i^{(m)}}. \end{aligned}$$

**Example 4.1.** Let us consider the autonomous problem proposed in [6, 10] ( $n = 1, m = 2$ ):  $\mathcal{L}(s, v, w) = |s^2 - v^5|^2 |w|^{22} + \varepsilon |w|^2$ ,  $t \in [0, 1]$ . The problem satisfies hypotheses (H1)-(H3) of Tonelli's existence theorem. Function  $\tilde{x}(t) = kt^{\frac{5}{3}}$  verifies the integral form of the Euler-Lagrange equations (10). However,  $\tilde{x}$  belongs to  $W_2^2$  but not to  $W_2^\infty$ . The regularity condition (9) of Theorem 4.1 is not satisfied.

*Proof.* The proof is by contradiction and is analogous to that of Theorem 3.1. Suppose that (10) is not satisfied. For  $i = 1, \dots, n$  and  $|\alpha| \leq 1$ , we consider the curve  $C_\alpha : t = t_\alpha(s)$ ,  $x = X_\alpha(s)$ ,  $0 \leq s \leq l$ , with

$$\begin{aligned} X_{i\alpha}(s) &= X_i(s) + \sum_{i=1}^m \alpha^i \psi^{(i-1)}(s), \\ X_{j\alpha}(s) &= X_j(s), \quad j \neq i, \end{aligned}$$

$\psi$  defined as in the proof of Theorem 3.1. We have  $|\psi^{(m)}(s)| \leq l$  a.e and we choose  $\alpha$  small enough in order to verify

$$\left| X_{i\alpha}^{(k)}(s) - X_i^{(k)}(s) \right| \leq \delta.$$

Thus,  $J[C] \leq J[C_\alpha]$  for all  $|\alpha| \leq \alpha_0$ . Setting

$$\phi(\alpha, s) = F(t(s), X(s), t'(s), X'(s), \dots, t^{(m)}(s), X^{(m)}(s))$$

we have

$$\left. \frac{\partial \phi}{\partial \alpha} \right|_{\alpha=0} = \frac{\partial F}{\partial x_i} \psi + \frac{\partial F}{\partial \dot{x}_i} \psi' + \dots + \frac{\partial F}{\partial x_i^{(m)}} \psi^{(m)}, \text{ for } s \in [0, l] \quad a.e.$$

By the hypotheses  $(S_i)_{0 \leq i \leq n}$  we have for  $\alpha \leq \alpha_0$  that  $\frac{\partial \phi}{\partial \alpha}$  is, in absolute value, bounded in  $E_1 \cup E_2$  by a  $L$ -integrable function in  $[0, l]$ . The proof continues in the same lines as in the end of the proof of Theorem 3.1, applying the usual rule of differentiation under the integral sign and integration by parts, which leads to a contradiction.  $\square$

## 5 Conclusions

The search for appropriate conditions on the data of the problems of the calculus of variations with higher-order derivatives, under which we have regularity of solutions or under which more general necessary conditions hold, is an important area of study. In this paper we generalize our previous results [21] to problems of the calculus of variations of higher order than two. We have proved duBois-Reymond and Euler-Lagrange type necessary optimality conditions valid in the class of functions where the existence is proved. Minimizers in this class may have unbounded derivatives and fail to satisfy the classical necessary conditions of duBois-Reymond or Euler-Lagrange. We prove that if the derivatives of the Lagrangian function with respect to the highest derivatives are superlinear or coercive, then all the minimizers have essentially bounded derivatives. This imply non-occurrence of the Lavrentiev phenomenon and validity of classical necessary optimality conditions.

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